Lorentz invariants (scalars)

$$F^{\mu\nu}F_{\mu\nu} = 2(\vec{E}^2 - \vec{B}^2)$$

$$F^{\mu\nu}\tilde{F}_{\mu\nu} = -4\vec{E}\cdot\vec{B}$$

where

$$\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$$

which means that

$$\vec{E}'^2 - \vec{B}'^2 = \vec{E}^2 - \vec{B}^2$$

$$\vec{E}'\cdot\vec{B}' = \vec{E}\cdot\vec{B}$$

<u>remark</u>:

$$\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} = -F^{\mu\nu}F_{\mu\nu}$$

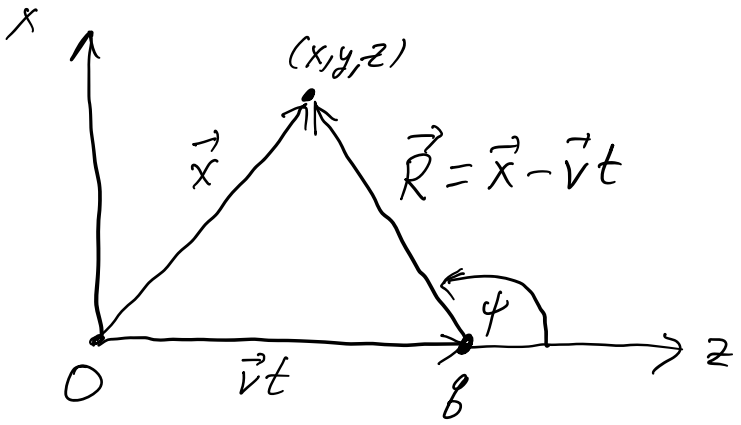
<u>Application</u>: The electromagnetic fields of a uniformly moving charge

Reference frame

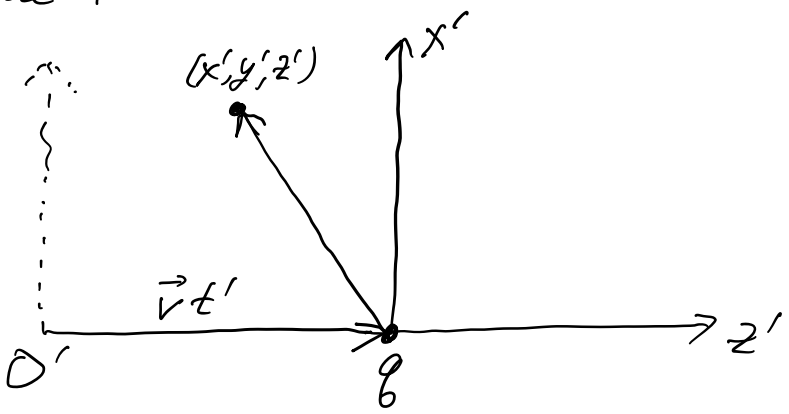
K (lab frame) - observe a charge moving with velocity $\vec{v} = \vec{\beta}c$

K' (rest frame of the charge q)

Reference frame K $\quad (\vec{v} = v\hat{z})$



Reference frame K'



Note: at $t = t' = 0$, the origins of K, K' coincide.

In K', $\quad \vec{E}' = \dfrac{q\vec{x}'}{r'^3}$, $\quad \vec{B}' = 0$

$r' = |\vec{x}'|$

In K, $\quad \vec{E}_\perp = \gamma(\vec{E}'_\perp - \vec{\beta} \times \vec{B}'_\perp)$, $\quad \vec{E}_{\parallel} = \vec{E}'_{\parallel}$

$B_\perp = \gamma(B'_\perp + \vec{\beta} \times \vec{E}'_\perp)$, $\quad \vec{B}_{\parallel} = \vec{B}'_{\parallel}$

We know that

$$\vec{E}'_{\parallel} = \frac{q\vec{x}'_{\parallel}}{r'^3}, \qquad \vec{E}'_{\perp} = \frac{q\vec{x}'_{\perp}}{r'^3}$$

$$\vec{E} = \vec{E}_{\parallel} + \vec{E}_{\perp} = \vec{E}'_{\parallel} + \gamma\vec{E}'_{\perp} = \frac{q}{r'^3}(\vec{x}'_{\parallel} + \gamma\vec{x}'_{\perp})$$

$$\vec{B} = \vec{B}_{\parallel} + \vec{B}_{\perp} = \gamma\vec{\beta}\times\vec{E}'_{\perp} = \frac{\gamma q}{r'^3}\vec{\beta}\times\vec{x}'_{\perp}$$

$\Rightarrow$

$$\vec{E} = \frac{\gamma q}{r'^3}(\vec{x}_{\parallel} - \vec{v}t + \vec{x}_{\perp})$$

$$\vec{B} = \frac{\gamma q}{cr'^3}\vec{v}\times\vec{x}_{\perp}$$

$$r'^2 = \gamma^2(\vec{x}_{\parallel} - \vec{v}t)\cdot(\vec{x}_{\parallel} - \vec{v}t) + \vec{x}_{\perp}\cdot\vec{x}_{\perp}$$

$$= \gamma^2 R_{\parallel}^2 + R_{\perp}^2$$

where $\vec{R}_{\parallel} = \vec{x}_{\parallel} - \vec{v}t$, $\qquad \vec{R}_{\perp} = \vec{x}_{\perp}$

$$\vec{E} = \frac{\gamma q\vec{R}}{(\gamma^2 R_{\parallel}^2 + R_{\perp}^2)^{3/2}}, \qquad \vec{B} = \frac{\gamma q\vec{v}\times\vec{R}}{c(\gamma^2 R_{\parallel}^2 + R_{\perp}^2)^{3/2}}$$

Note

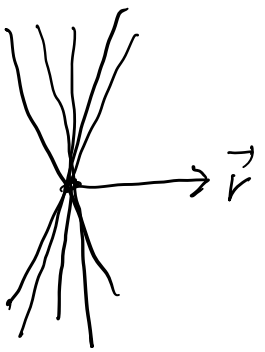
$$(\gamma^2 R_{\parallel}^2 + R_{\perp}^2)^{3/2} = \gamma^3(R_{\parallel}^2 + R_{\perp}^2(1-\beta^2))^{3/2}$$

$$= \gamma^3 R^3(1 - \beta^2\sin^2\psi)^{3/2}$$

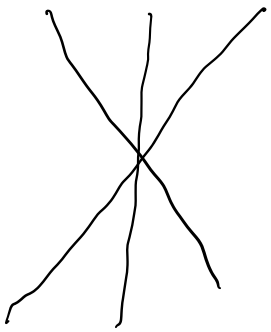
since $R_\perp = R\sin\psi$. Thus

$$\vec{E}(\vec{x}) = \frac{q\vec{R}}{\gamma^2 R^3 (1-\beta^2 \sin^2\psi)^{3/2}}$$

$$\vec{B}(\vec{x}) = \frac{q\vec{v}\times\vec{R}}{c\gamma^2 R^3 (1-\beta^2 \sin^2\psi)^{3/2}}$$



K



K'

Electromagnetic Radiation

In the Lorenz gauge

$$\Box \vec{A} = \frac{4\pi}{c}\vec{J} \qquad \Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

$$\Box \Phi = 4\pi\rho$$

$$\Box A^\mu = \frac{4\pi}{c} J^\mu$$

Let us examine

$$\Box \psi(\vec{x},t) = 4\pi f(\vec{x},t)$$

General solution

$$\psi(\vec{x},t) = \psi_0(\vec{x},t) + \psi_1(\vec{x},t)$$

where

$$\Box \psi_0(\vec{x},t) = 0$$

ψ_0 is fixed by initial conditions

But, ψ_1 is not unique

Roughly, $\psi(\vec{x},t) = 4\pi \Box^{-1} f(\vec{x},t)$

Note that $\Box$ has a nontrivial eigenfunction with zero eigenvalue.

If $A\psi = 0$ $\qquad$ $\det A = \prod_i \lambda_i = 0$

A^{-1} does not exist.

$A\psi = f \Rightarrow$

solution ψ either does not exist or exists but is not unique.

We will see that the solution to the inhomogeneous equation, $\psi_1(\vec{x},t)$ is non-unique.

$\psi_1(\vec{x},t)$ will be determined by imposing a physical condition: causality.

Green function technique

$$\Box_x G(x,x') = 4\pi \delta^4(x-x')$$

$$\delta^4(x-x') = \delta^3(\vec{x}-\vec{x}')\,\delta(x_0-x_0') \qquad x_0 = ct$$

By translational invariance, $G(x,x') = G(x-x')$

Sufficient to solve

$$\Box G(x) = 4\pi \delta^4(x)$$

Once you know $G(x)$, then

$$\psi_1(x) = \int d^4x'\, G(x-x')\, f(x')$$

because

$$\Box \psi_1(x) = 4\pi \int d^4x'\, \delta^4(x-x')\, f(x') = 4\pi f(x)$$

✓

Method of solution: Fourier transform

(converts a differential equation into an algebraic equation)

$$G(x) = \frac{1}{(2\pi)^4}\int d^4k\ \tilde{G}(k)\, e^{-ik\cdot x}$$

$$\left(k\cdot x = k_0 x_0 - \vec{k}\cdot\vec{x}\right)$$

$$\delta^4(x) = \frac{1}{(2\pi)^4}\int d^4x\, e^{-ik\cdot x}$$

Plug these into $\Box G(x) = 4\pi\delta^4(x)$

$$\Longrightarrow \qquad -k^2\tilde{G}(k) = 4\pi$$

$$\tilde{G}(k) = -\frac{4\pi}{k^2}, \qquad k^2 = k_0^2 - |\vec{k}|^2$$

$$G(x) = -\frac{4\pi}{(2\pi)^4}\int \frac{d^4k}{k^2}\, e^{-ik\cdot x}$$

$$= -\frac{1}{4\pi^3}\int d^3k\, e^{i\vec{k}\cdot\vec{x}} \int_{-\infty}^{\infty} dk_0\, \frac{e^{-ik_0 x_0}}{k_0^2 - |\vec{k}|^2}$$

Problem!! singularity at $k_0 = \pm|\vec{k}|$

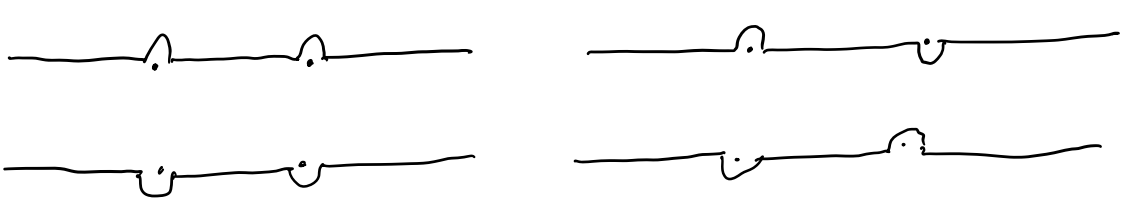
$\Rightarrow$ integral over k_0 does not exist!

Even though $\Box^{-1}$ does not exist, we need a strategy to solve $\Box\psi = 4\pi f$

[or to solve $\Box G = 4\pi\delta^4(x)$]

Strategy: deform the integration over k_0 (infinitesimal deformation) so that integral is well-defined.

Deform the contour by an excursion into the complex k_0 plane



Impose causality $\Rightarrow$

"chooses"

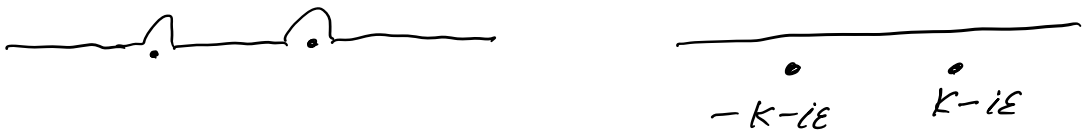
$G_R(x,t) = 0$ if $t < 0$ R="retarded"

Equivalently

$$G_R(x) = \frac{-1}{4\pi^3}\int d^3k\, e^{i\vec{k}\cdot\vec{x}} \int_{-\infty}^{\infty} dk_0 \, \frac{e^{-ik_0 x_0}}{(k_0 - K + i\varepsilon)(k_0 + K + i\varepsilon)}$$

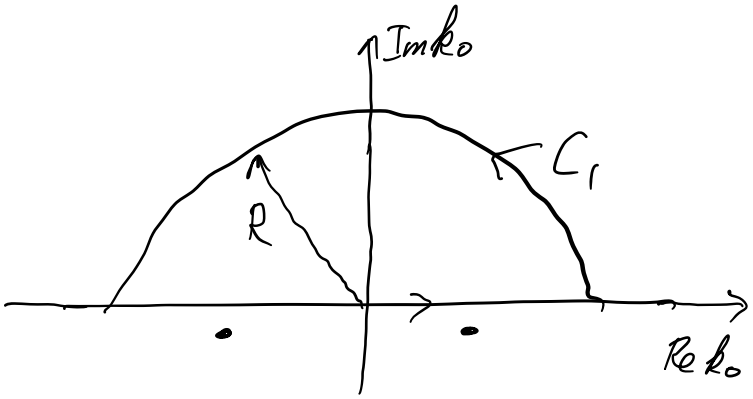
where $K \equiv |\vec{k}|$. $\quad \varepsilon > 0$ infinitesimal

poles appear at $k_0 = \pm K - i\varepsilon \quad (\mathrm{Im}\, k_0 < 0)$



1. Suppose $x_0 = t < 0$

$|x_0| = -x_0$



$(R \to \infty)$

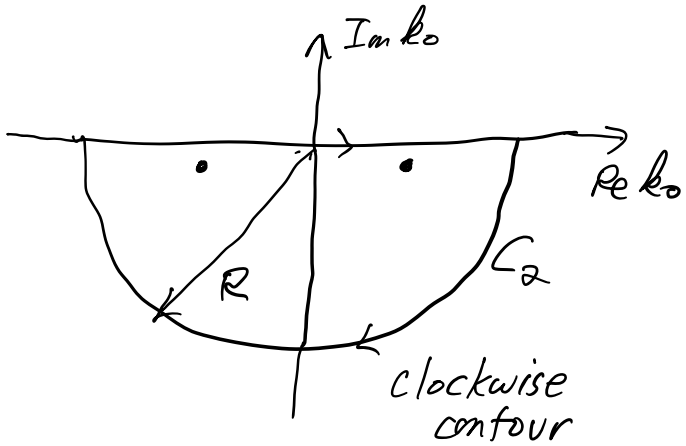
Hence,

$$G_R(x) = 0 \quad \text{if } t < 0$$

2. Suppose $x_0 = t > 0$

Close contour in lower half k_0 plane.

$|x_0| = x_0$



$$\int_{C_2} dk_0 \; \frac{e^{-ik_0|x_0|}}{(k_0 - K + i\varepsilon)(k_0 + K + i\varepsilon)}$$

$$= \frac{-2\pi i}{2K}\left[e^{-iKx_0} - e^{iKx_0}\right] \qquad \text{after taking } \varepsilon \to 0$$

$$= -\frac{2\pi}{K}\sin(Kx_0)$$

Hence,

$$G_R(x) = \frac{1}{2\pi^2}\,\theta(x_0)\int d^3k\, e^{i\vec{k}\cdot\vec{x}}\,\frac{\sin Kx_0}{K}$$

$$= \frac{1}{\pi}\,\theta(x_0)\int_0^\infty K\,dK\,\sin Kx_0 \int_{-1}^{1} d\cos\theta\, e^{ik|\vec{x}|\cos\theta}$$

where $d^3k = K^2\,dK\,d\cos\theta\,d\phi$

$$G_R(x) = \frac{1}{\pi}\theta(x_0)\int_0^\infty K\,dK \sin Kx_0 \,\frac{1}{iK|\vec{x}|}\left[e^{iK|\vec{x}|} - e^{-iK|\vec{x}|}\right]$$

$$= \frac{2}{\pi r}\theta(x_0)\int_0^\infty dK \sin Kx_0 \sin Kr$$

$$r \equiv |\vec{x}|$$

$$= \frac{\theta(x_0)}{2\pi r}\int_{-\infty}^{\infty} dK\left[e^{i(x_0-r)K} - e^{i(x_0+r)K}\right]$$

$$= \frac{\theta(x_0)}{r}\left[\delta(x_0-r) + \delta(x_0+r)\right]$$

$$= \frac{\theta(x_0)}{r}\,\delta(x_0-r)$$

Thus,

$$\boxed{G_R(x) = \frac{\theta(x_0)\,\delta(x_0-|\vec{x}|)}{|\vec{x}|}}$$

You don't need the $\theta(x_0)$, since $\delta(x_0-|\vec{x}|)=0$ if $x_0<0$.

$$\theta(x_0) = \begin{cases} 1, & x_0 > 0 \\ 0, & x_0 < 0 \end{cases}$$

Last time, we outlined a method to solve

$$\Box \psi(\vec{x},t) = 4\pi f(\vec{x},t).$$

The solution to this inhomogeneous wave equation is

$$\psi(\vec{x},t) = \psi_0(\vec{x},t) + \psi_1(\vec{x},t)$$

↑ "particular solution"

where $\Box \psi_0(\vec{x},t) = 0$

and

$$\psi_1(x) = \int d^4x' \, G_R(x-x') f(x')$$

where $x \equiv (ct; \vec{x})$ and

$$\Box_x G_R(x,x') = 4\pi \delta^4(x-x')$$

Imposing causality, i.e. $G_R(x,t) = 0$ if $t<0$, where $G_R(x,x') = G_R(x-x')$, we found that

$$G_R(x) = \frac{\delta(x_0 - |\vec{x}|)}{|\vec{x}|}$$

retarded Green function

Hence,

$$\psi_1(x) = \int d^3x' \int_{-\infty}^{\infty} dx_0' \, \frac{\delta(x_0 - x_0' - |\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|} f(\vec{x}', t')$$

$(x_0' = ct')$

$$= \int \frac{d^3x'}{|\vec{x}-\vec{x}'|} f\left(x', t' = t - \frac{|\vec{x}-\vec{x}'|}{c}\right)$$

Definition: (t' is the "retarded time")

$$[f(\vec{x}', t')]_{ret} \equiv f\left(\vec{x}', t' = t - \frac{|\vec{x} - \vec{x}'|}{c}\right)$$

$$\boxed{\Psi_i(x) = \int d^3x' \, \frac{[f(x', t')]_{ret}}{|\vec{x} - \vec{x}'|}}$$

$$t' = t - \frac{|\vec{x} - \vec{x}'|}{c}$$

For electromagnetic waves, (Lorenz gauge)

$$\Box A^\mu = \frac{4\pi}{c} J^\mu \qquad J^\mu(c\rho; \vec{J})$$

$$\vec{A}(\vec{x}, t) = \frac{1}{c} \int d^3x' \, \frac{[\vec{J}(\vec{x}', t')]_{ret}}{|\vec{x} - \vec{x}'|}$$

$$\Phi(\vec{x}, t) = \int d^3x' \, \frac{[\rho(\vec{x}', t')]_{ret}}{|\vec{x} - \vec{x}'|}$$

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \qquad \text{(in cgs units)}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Since $A^\mu = (\Phi; \vec{A})$

$$\Box \vec{E} = -4\pi\left(\vec{\nabla}\rho + \frac{1}{c^2}\frac{\partial \vec{J}}{\partial t}\right)$$

$$\Box \vec{B} = \frac{4\pi}{c}\vec{\nabla}\times\vec{J}$$

Hence,

$$\vec{E}(\vec{x},t) = -\int \frac{d^3x'}{|\vec{x}-\vec{x}'|}\left[\vec{\nabla}'\rho + \frac{1}{c^2}\frac{\partial \vec{J}}{\partial t'}\right]_{ret}$$

$$\vec{B}(\vec{x},t) = \frac{1}{c}\int \frac{d^3x'}{|\vec{x}-\vec{x}'|}\left[\vec{\nabla}'\times\vec{J}\right]_{ret}$$

Note:

$$[f(\vec{x}',t')]_{ret} = f\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)$$

In particular,

$$\vec{\nabla}'[f]_{ret} \neq [\vec{\nabla}' f]_{ret}$$

$\uparrow$ fixed t $\qquad$ $\uparrow$ fix t'

(and $\vec{x}$ is fixed).

Use the chain rule!